Analysis and Synthesis of Anisochronic Systems – a Survey

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- Traditional time delay systems with delayed input (TDS)
- Anisochronic systems and models – description (AS)
- The spectrum and the stability of AS
- Identification and modeling of AS
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- Algebraic controller design in R_{MS} ring
- Controller tuning method: Pole assignment
- Application examples
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Time delay systems

Time delay systems with delayed input (TDS)

- Example:

$$y''(t) + 3y'(t) + 2y(t) = u'(t-\tau) + 3u(t-\tau)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}\exp(-\tau s)$$

- LUMPED delay (difference-differential equations)
- Transport processes, population systems in biology, …

Time delay systems 2/2

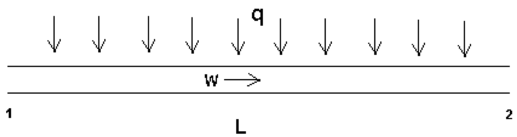
DISTRIBUTED delays (partial differential equations)

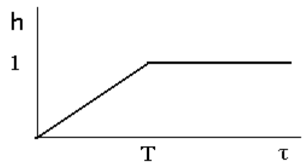
- Long lines (electrical, heating), heat exchanger,…
- Can be approximated by Stieltjes integrals => lumped delays

$$y(t)=\int_0^T u(t-\tau)dh(\tau)=\int_0^T u(t-\tau)\bar{h}'(\tau)d\tau+\sum_{i=1}^{m}\Delta h_i^* u(t-\tau_i)$$

$$\int_0^T u(t-\tau)h'(\tau)d\tau \overset{Laplace}{\rightarrow} u(s)\int_0^T \exp(-s\tau)h'(\tau)d\tau$$

- Example:





$$\vartheta_2(t)=\vartheta_1(t-T)+\frac{1}{c\rho}\int_0^T q(t-\tau)d\tau;\quad T=\frac{L}{w}$$

$$\int_0^T q(t-\tau)d\tau \overset{L}{\rightarrow} q(s)\int_0^T \exp(-s\tau)d\tau = q(s)\frac{1}{s}\left[1-\exp(-Ts)\right]\overset{L^{-1}}{\rightarrow}\int_0^t\left[q(\eta)-q(\eta-\tau)\right]d\eta$$

Anisochronic systems (AS)

Anisochronic models of retarded type (SISO case)

$$\frac{d\mathbf{x}(t)}{dt} = \int_0^T d\mathbf{A}(\tau)\mathbf{x}(t-\tau)d\tau + \int_0^T d\mathbf{B}(\tau)\mathbf{u}(t-\tau)d\tau = \mathbf{A}_0\mathbf{x}(t) + \sum_{i=1}^{l}\mathbf{A}_i\mathbf{x}(t-\vartheta_i) + \mathbf{b}_0 u(t) + \sum_{j=1}^{k}\mathbf{b}_j u(t-\tau_j)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

$$s\mathbf{x}(s) = \left[\int_0^L \exp(-s\tau)d\mathbf{A}(\tau)\right]\mathbf{x}(s) + \left[\int_0^L \exp(-s\tau)d\mathbf{B}(\tau)\right]u(s)$$

$$= \left[\mathbf{A}_0 + \sum_{i=1}^{l}\mathbf{A}_i \exp(-s\vartheta_i)\right]\mathbf{x}(s) + \left[\mathbf{B}_0 + \sum_{j=1}^{k}\mathbf{B}_i \exp(-s\tau_j)\right]u(s)$$

$$y(s) = \mathbf{C}\mathbf{x}(s)$$

Anisochronic systems (AS)

2/2

Corresponding transfer function

$$G(s)=\frac{N(s)}{M(s)}=\frac{Y(s)}{U(s)}=\frac{\mathbf{C}\operatorname{adj}\left[s\mathbf{I}-\left[\int_0^L \exp(-s\tau)d\mathbf{A}(\tau)\right]\right]\left[\int_0^L \exp(-s\tau)d\mathbf{B}(\tau)\right]}{\det\left[s\mathbf{I}-\left[\int_0^T \exp(-s\tau)d\mathbf{A}(\tau)\right]\right]}=$$

$$=\frac{\mathbf{C}\operatorname{adj}\left[s\mathbf{I}-\mathbf{A}_0-\sum_{i=1}^{l}\mathbf{A}_i\exp(-s\vartheta_i)\right]\left[\mathbf{b}_0+\sum_{j=1}^{k}\mathbf{b}_j\exp(-s\tau_j)\right]}{\det\left[s\mathbf{I}-\mathbf{A}_0-\sum_{i=1}^{l}\mathbf{A}_i\exp(-s\vartheta_i)\right]}$$

Examples

$$G(s)=\frac{K\exp(-\tau s)}{Ts+\exp(-\vartheta s)} \qquad G(s)=\frac{K\left[T_B s+\exp(-\psi s)\right]}{\left[T_1 s+1\right]\left[Ts+\exp(-\vartheta s)\right]}\exp(-\tau s)$$

Spectrum and stability of AS 1/2

Characteristic quasipolynomial

$$M(s)=\det\left[s\mathbf{I}-\left[\int_0^T \exp(-s\tau)d\mathbf{A}(\tau)\right]\right]=\det\left[s\mathbf{I}-\mathbf{A}_0-\sum_{i=1}^{l}\mathbf{A}_i\exp(-s\vartheta_i)\right]$$

Argument principle

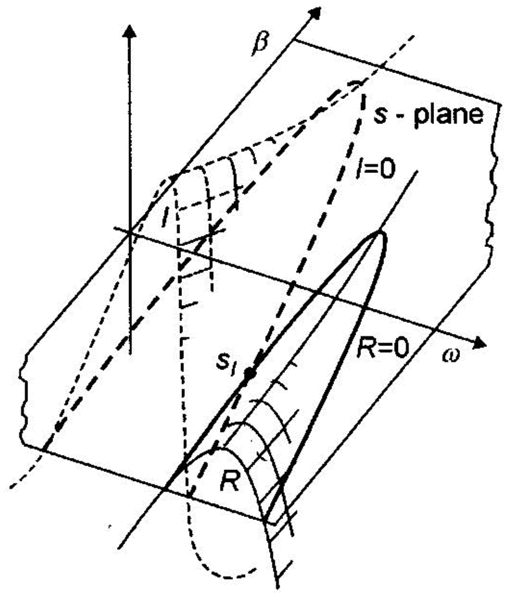
$$N_D=\frac{1}{2\pi \mathrm{j}}\int_{\varphi^+}\frac{M'(s)}{M(s)}ds=\frac{1}{2\pi}\Delta_{\varphi^+}\arg M(s)$$

$$N_U=\frac{n}{2}-\frac{1}{\pi}\underset{s=\omega \mathrm{j},\omega\in[0,\infty]}{\Delta\arg M(s)}\overset{N_U=0}{\Rightarrow}\underset{s=\omega \mathrm{j},\omega\in[0,\infty]}{\Delta\arg M(s)}=n\frac{\pi}{2}$$

Spectrum and stability of AS 2/2

Mapping based rootfinder (Vyhlídal, 2003)

1. Region D, cover by equidistant nodes
2. For each node compute $R(\alpha+\omega\mathrm{j}), I(\alpha+\omega\mathrm{j})$
3. Map the intersections $R(\alpha+\omega\mathrm{j})=0, I(\alpha+\omega\mathrm{j})=0$
4. Both intersections => approximation of poles
5. Enhanced by Newton's iteration method



$$M(\alpha+\omega\mathrm{j})=R(\alpha+\omega\mathrm{j})+\mathrm{j}I(\alpha+\omega\mathrm{j})$$

- Weyl's algorithm
-Discretization of solution operator
- …

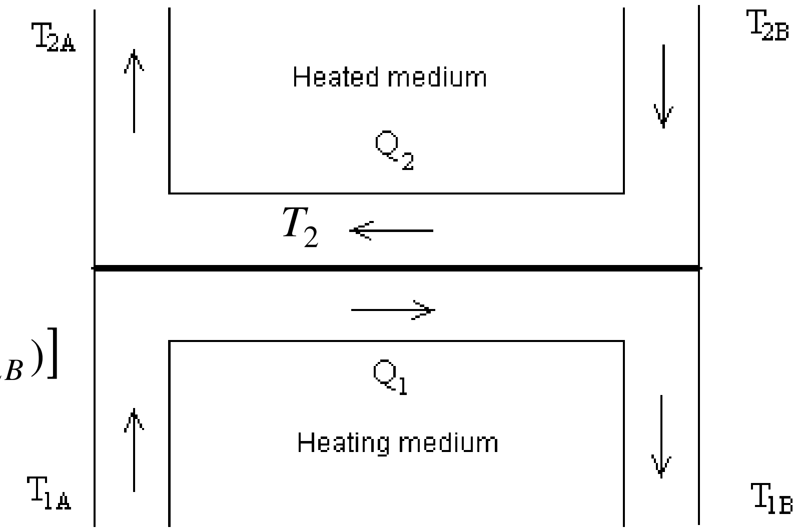
Modeling of AS

Example: Heat exchanger

Heat transfer

$$W = k\Delta\overline{T};\quad \Delta\overline{T} \cong T_{1A} - 0.5(1+q)T_{2A} - 0.5(1-q)T_{2B}$$

$$q = \frac{Q_2}{Q_1}$$



System dynamics

$$\frac{dT_2(t)}{dt} = \frac{k}{C_1}\left[T_{1A}(t-\tau_{1A}) - \alpha T_{2A}(t-\tau_{2A}) - \beta T_{2B}(t-\tau_{2B})\right] - \frac{Q_2 c}{C_1}\left[T_{2A}(t) - T_{2B}(t)\right]$$

$$\vartheta_S \frac{dT_{2A}(t)}{dt} = T_2(t) - T_{2A}(t)$$

Modeling of AS

Example: Heat exchanger

$$\mathbf{u}=\begin{bmatrix}T_{1A}\\ T_{2B}\end{bmatrix};\quad \mathbf{x}=\begin{bmatrix}T_{2}\\ T_{2A}\end{bmatrix};\quad y=T_{2A}$$

$$\dot{\mathbf{x}}=\mathbf{A}(s)\mathbf{x}+\mathbf{B}(s)\mathbf{u}$$

$$y=\mathbf{C}(s)\mathbf{x}$$

$$\mathbf{A}(s)=\begin{bmatrix}0 & -\frac{1}{C_1}\left[\alpha K\exp(-\tau_{2A}s)+Q_2c\right]\\ \frac{1}{T_S} & \frac{-1}{T_S}\end{bmatrix}$$

$$\mathbf{B}(s)=\begin{bmatrix}\frac{K}{C_1}\exp(-\tau_{1A}s) & \frac{1}{C_1}\left[Q_2c-\beta K\exp(-\tau_{2B}s)\right]\\ 0 & 0\end{bmatrix}$$

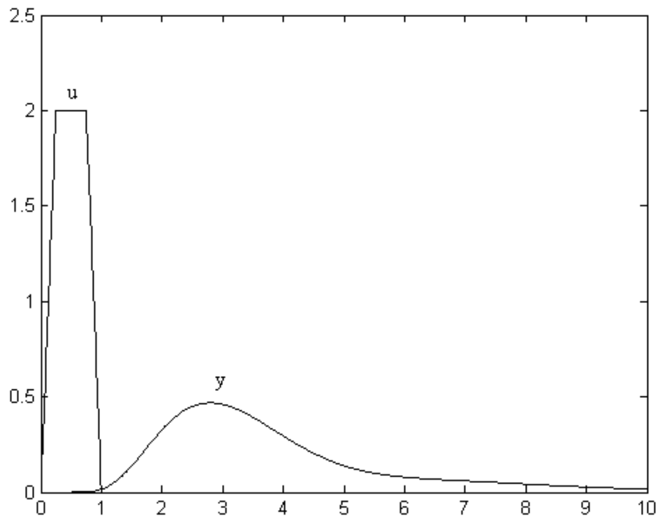
$$\mathbf{C}(s)=\begin{bmatrix}0 & 1\end{bmatrix}$$

$$G(s)=\frac{\mathbf{C}(s)\left[s\mathbf{I}-\mathbf{A}(s)\right]\mathbf{B}(s)}{\det\left(s\mathbf{I}-\mathbf{A}(s)\right)}$$

Model identification

1. Successive integrations

a) Integrable input



$$Ty'(t) + y(t-\vartheta) = Ku(t-\tau)$$

$$y_I(t) = \int_0^t y(\sigma)d\sigma; u_I(t) = \int_0^t u(\sigma)d\sigma \Rightarrow \lim t \to \infty$$

$$y_{II}(t) = \int_0^t [y_I(\infty) - y_I(\sigma)]d\sigma; u_{II}(t) = \int_0^t [u_I(\infty) - u_I(\sigma)]d\sigma$$

...

$$T = (0.95 \div 1)T_S; \quad K = \frac{y_I(\infty)}{u_I(\infty)}$$

$$(T-\vartheta)y_I(\infty) - y_I(\infty) = -K[\tau u_I(\infty) + u_I(\infty)]$$

$$0.5\vartheta^2 y_I(\infty) - (T-\vartheta)y_{II}(\infty) + y_{III}(\infty) =$$
$$= K[0.5\tau^2 u_I(\infty) + \tau u_{II}(\infty) + u_{III}(\infty)]$$

Model identification

1. Successive integrations

b) Step input

$$Ty'(t) + y(t-\vartheta) = Ku(t-\tau)$$

$$y_I(t) = \int_0^t [y(\infty) - y(\sigma)]d\sigma; u_I(t) = \int_0^t [u(\infty) - u(\sigma)]d\sigma \Rightarrow \lim t \to \infty$$

$$y_{II}(t) = \int_0^t [y_I(\infty) - y_I(\sigma)]d\sigma; u_{II}(t) = \int_0^t [u_I(\infty) - u_I(\sigma)]d\sigma$$

...

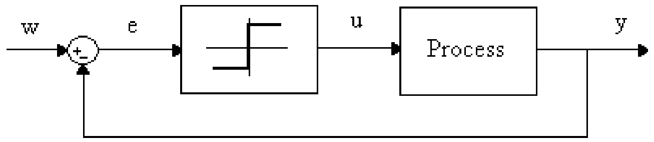
$$(T - \vartheta + \tau)y(\infty) = y_I(\infty)$$

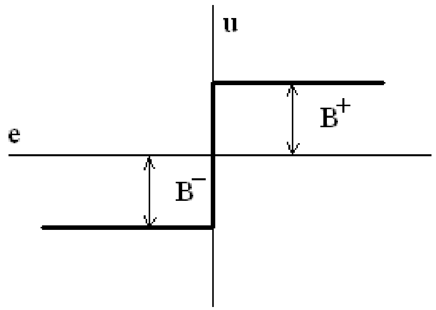
$$0.5(\vartheta^2 - \tau^2)y(\infty) - (T - \vartheta)y_I(\infty) = -y_{II}(\infty)$$

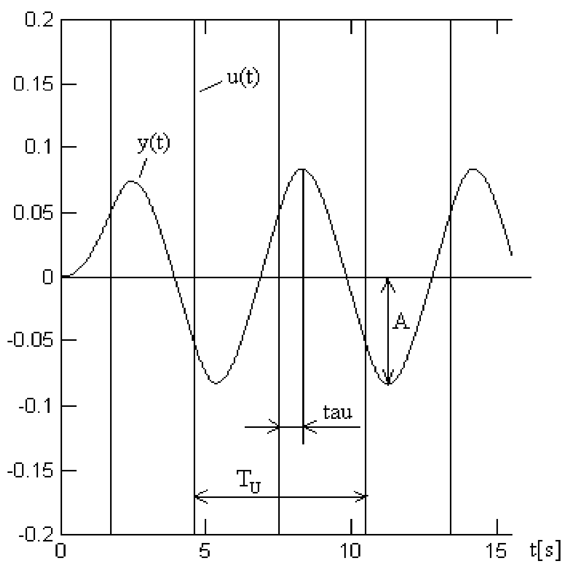
$$\frac{1}{6}(\vartheta^3 - \tau^3) - 0.5\vartheta^2 y_I(\infty) + (T - \vartheta)y_{II}(\infty) = y_{III}(\infty)$$

Model identification

2. Relay in the feedback







$$R(A)G(j\omega_u) = -1 + 0j$$

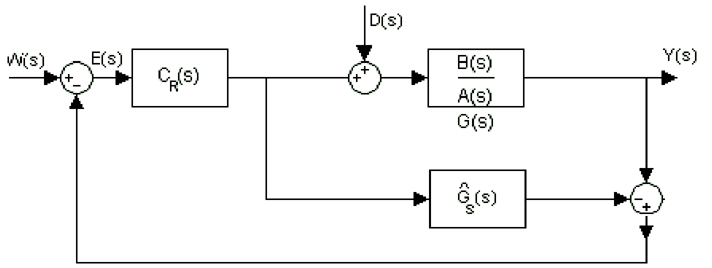
$$|R(A)G(j\omega_u)| = 1$$

$$\arg[R(A)G(j\omega_u)] = -\pi$$

$$R(A) = \frac{4B}{\pi A} \qquad K = \frac{\int_0^{T_u} y(t)dt}{\int_0^{T_u} u(t)dt}; \; T_u = \frac{2\pi}{\omega_u}$$

Control methods

1. Generalized internal model control (IMC) principle

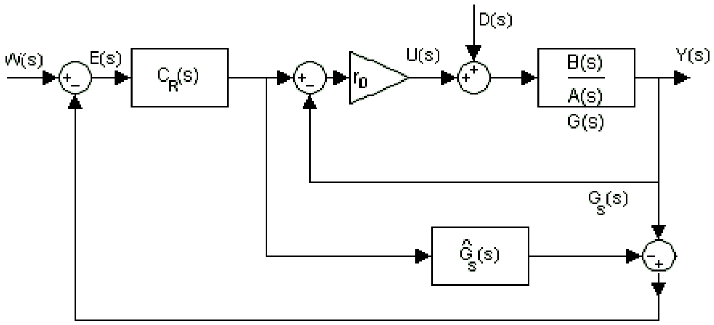


$$G(s) = \hat{G}(s) \quad G(s) = \frac{B(s)}{A(s)} \quad =>$$

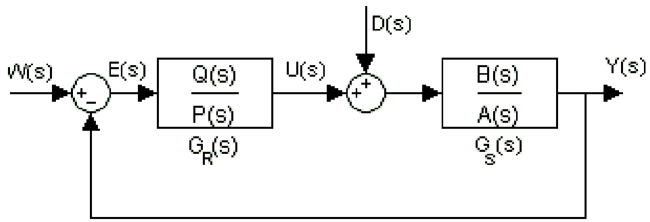
$$G_{WY}(s) = \frac{Y(s)}{W(s)} = C_R(s) G_S(s)$$

$$G_S(s) = G_{S0}(s) D(s); \quad D(0) = 1$$

$$C_R(s) = G_{S0}^{-1}(s) F(s); \quad F(0) = 1$$



$$\lim_{s \to 0} [G_R(s)]^{-1} = 0$$



$$G_R(s) = \frac{C_R(s)}{1 - C_R(s) G_S(s)}$$

Control methods

2. Algebraic control in R_{MS} ring

- R_{MS} = ring of stable and proper retarded quasipolynomial (RQ) meromorphic functions
- Holomorphic vs. meromorphic functions

$$f_1(s) = \frac{s^2 + s - 1}{s^3 + 2s + 3} \qquad f_2(s) = \frac{\sin(s)}{s + \exp(-s)}$$

- RQ-meromorphic functions: description of a general term in R_{MS}

$$T(s) = \frac{y(s)}{x(s)} = \frac{y_0(s)\exp(-\tau s)}{x(s)}$$

where $y_0(s)$ is a (quasi)polynomial

$x(s)$ is a stable (quasi)polynomial

τ is non-negative

deg $x(s)$ is greater than or equal to deg $y(s)$ => Properness

Control methods

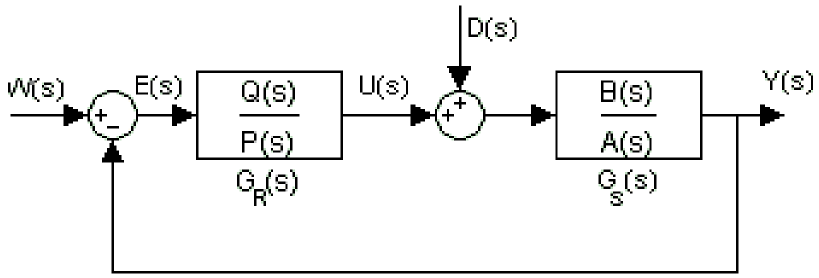
- Examples of a models in R_{MS}

$$G_1(s) = \frac{K\exp(-\tau s)}{(T_1 s + 1)[Ts + \exp(-\vartheta s)]} = \frac{\dfrac{K\exp(-\tau s)}{(s+m)^2}}{\dfrac{(T_1 s + 1)[Ts + \exp(-\vartheta s)]}{(s+m)^2}}$$

$$G_2(s) = \frac{K\exp(-\tau s)}{T_1 s - \exp(-\vartheta s)} = \frac{\dfrac{K\exp(-\tau s)}{Km\exp(-\tau s) + T_1 s - \exp(-\vartheta s)}}{\dfrac{T_1 s - \exp(-\vartheta s)}{Km\exp(-\tau s) + T_1 s - \exp(-\vartheta s)}}$$

$$G_3(s) = \frac{B(s)}{A(s)} = \frac{\dfrac{b(s)}{m(s)}}{\dfrac{a(s)}{m(s)}} = \frac{\dfrac{K\exp(-\tau s)}{Ts - \exp(-\vartheta s) + r_0 K\exp(-\tau s)}}{\dfrac{Ts - \exp(-\vartheta s)}{Ts - \exp(-\vartheta s) + r_0 K\exp(-\tau s)}}$$

Control methods



$$G_S(s) = \frac{B(s)}{A(s)}; \quad G_R(s) = \frac{Q(s)}{P(s)}$$

$$W(s) = \frac{H_W(s)}{F_W(s)}; D(s) = \frac{H_D(s)}{F_D(s)}$$

$$A(s), B(s), Q(s), P(s), H_W(s), F_W, H_D(s), F_D(s) \in R_{MS}$$

- Stabilization

$$A(s)P_0(s) + B(s)Q_0(s) = 1$$

- Youla- Kučera parameterization

$$G_R(s) = \frac{Q(s)}{P(s)} = \frac{Q_0(s) + A(s)T(s)}{P_0(s) - B(s)T(s)}; \quad T(s) \in R_{MS}$$

$$E(s) = E_{WE}(s)W(s) + E_{DE}(s)D(s)$$

$$\Rightarrow E(s) \in R_{MS}$$

Control methods

- Example

$$G_S(s)=\frac{K\exp(-\tau s)}{Ts+\exp(-\vartheta s)}=\frac{\dfrac{K\exp(-\tau s)}{s+m}}{\dfrac{Ts+\exp(-\vartheta s)}{s+m}};\quad W(s)=D(s)=\frac{\dfrac{k}{s+m}}{\dfrac{s}{s+m}}$$

$$Q_0(s)=1\Rightarrow P_0(s)=\frac{s+m-K\exp(-\tau s)}{Ts+\exp(-\vartheta s)}$$

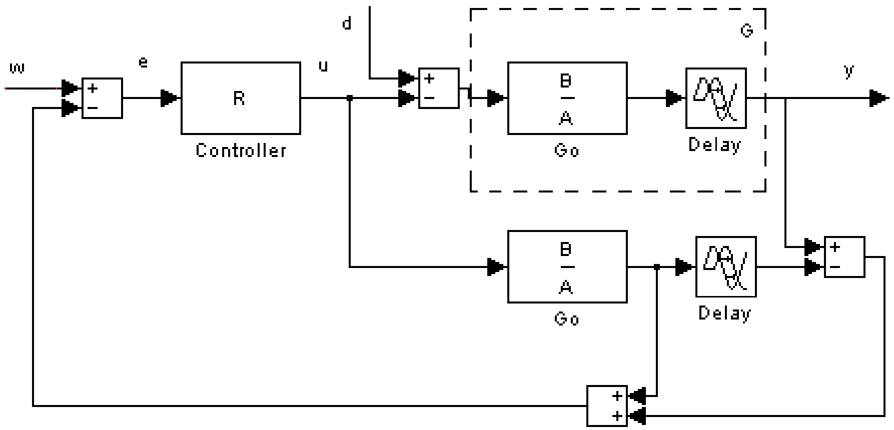
$$P(s)=\frac{s+m-K\exp(-\tau s)}{Ts+\exp(-\vartheta s)}-T(s)\frac{K\exp(-\tau s)}{s+m};\quad T(s)=\frac{\lambda(s+m)}{Ts+\exp(-\vartheta s)};\quad \lambda=\frac{m}{K}-1$$

$$\Rightarrow P(s)=\frac{s+m[1-\exp(-\tau s)]}{Ts+\exp(-\vartheta s)}$$

$$\Rightarrow Q(s)=\frac{m}{K}\quad\Rightarrow\quad G_R(s)=\frac{Q(s)}{P(s)}=\frac{m}{K}\frac{Ts+\exp(-\vartheta s)}{s+m[1-\exp(-\tau s)]}$$

Control methods

Comparison with Smith predictor



$$R(s) = \frac{G_R(s)}{1 + G_R(s)\left(G(s) - G_0(s)\right)}$$

Ad example:

$$G_R(s) = \frac{m}{K} \frac{Ts + \exp(-\vartheta s)}{s + m\left[1 - \exp(-\tau s)\right]}$$

$$R(s) = \frac{m}{K} \frac{Ts + \exp(-\vartheta s)}{s}$$

Tuning method

Pole placement

- Direct prescription of poles (Zítek)

$$m(\sigma_i, \mathbf{v}) = 0 = m(\sigma_i) + \sum_{j=1}^{n} v_j \left[\frac{\partial m(s, \mathbf{v})}{\partial v_j} \right]_{s=\sigma_i} ; \; i = 1 \ldots k$$

$$\operatorname{Re}[m(\sigma_i, \mathbf{v})] = 0$$

$$\operatorname{Im}[m(\sigma_i, \mathbf{v})] = 0$$

$$\mathbf{m} = \mathbf{S}\mathbf{v}$$

$$k = n \Rightarrow \mathbf{v} = \mathbf{S}^{-1}\mathbf{m}$$

$$k < n \Rightarrow \mathbf{v} = \mathbf{S}^{+}\mathbf{m}$$ Moore-Penrose inverse

Tuning method

Pole placement

- Continuous pole placement (Vyhlídal, Michiels)

$$\mathrm{Re}\left[m(\sigma_i+\Delta\sigma_i,\mathbf{v}+\Delta\mathbf{v})\right]=0=\mathrm{Re}\left[m(\sigma_i+\Delta\sigma_i,\mathbf{v})\right]+\sum_{j=1}^{n}\Delta v_j\left[\frac{\mathrm{Re}\left[\partial m(s,\mathbf{v})\right]}{\partial v_j}\right]_{s=\sigma_i+\Delta\sigma_i}$$

$$\mathrm{Im}\left[m(\sigma_i+\Delta\sigma_i,\mathbf{v}+\Delta\mathbf{v})\right]=0=\mathrm{Im}\left[m(\sigma_i+\Delta\sigma_i,\mathbf{v})\right]+\sum_{j=1}^{n}\Delta v_j\left[\frac{\mathrm{Im}\left[\partial m(s,\mathbf{v})\right]}{\partial v_j}\right]_{s=\sigma_i+\Delta\sigma_i}\quad ;\ i=1\ldots k$$

Algorithm:

1) $k = 1$

2) Compute the rightmost poles

3) Move k rightmost poles to the left

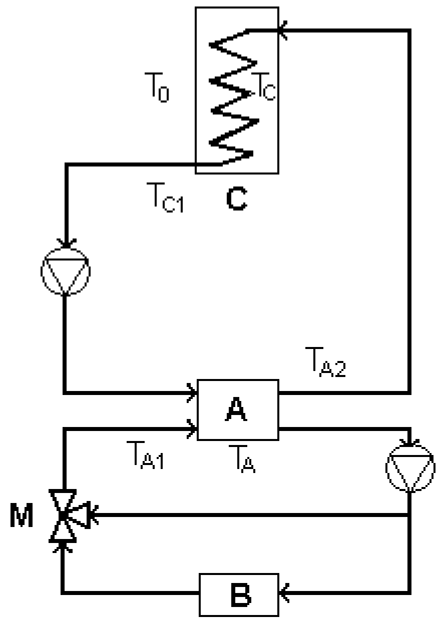
   a) Choose $\Delta\sigma_i \Rightarrow \Delta\mathbf{v}=\mathbf{S}^{+}\mathbf{m}$ (Vyhlídal)

   b) Calculate sensitivity functions $\left[\frac{\partial s}{\partial v}\right]_{\substack{s=\sigma_i\\ v=v_j}} \Rightarrow \Delta\mathbf{v}$ (Michiels)

4) Increase k or STOP (stability is reached, $k=n$, poles too close to each other,…)

Examples of applications

Model of heating system (CTU in Prague)



A – Plate heat exchanger

$$\vartheta_A \frac{dT_A(t)}{dt} = k_A\left[T_{A1}(t-\tau_{A1}) - \alpha T_{A2}(t-\tau_{MA}) - \beta T_{C1}(t-\tau_{CA}-\tau_1)\right]$$
$$+T_{C1}(t-\tau_{CA}) - T_{A2}(t)$$

$$\vartheta_{A2} \frac{dT_{A2}(t)}{dt} = T_A(t) - T_{A2}(t)$$

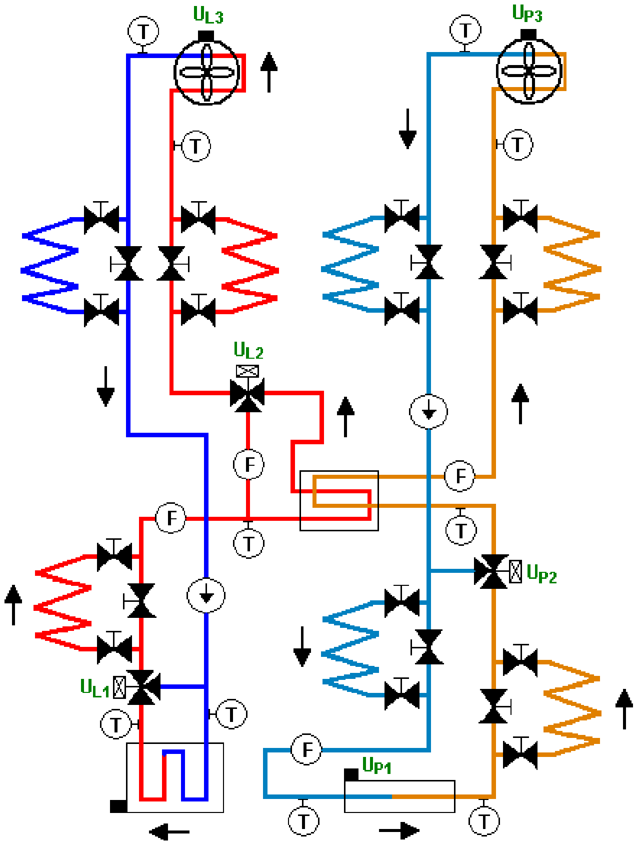
C – Cooler

$$\vartheta_{C1} \frac{dT_C(t)}{dt} = B(T_0)v(t-\tau_2) - k_2\left[\gamma T_{C1}(t-\tau_{C1}) + \varepsilon T_{A2}(t-\tau_{AC})\right]$$
$$-T_{C1}(t) + T_{A2}(t-\tau_{AC})$$

$$\vartheta_{C1} \frac{dT_{C1}(t)}{dt} = T_C(t) - T_{C1}(t)$$

Examples of applications

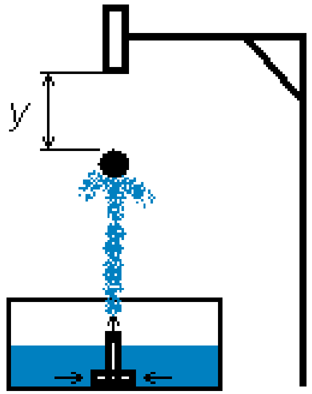
Model of heating system 2



- Two circuits
- Variable transport delay
- Controlled by PLC or PC

Examples of applications

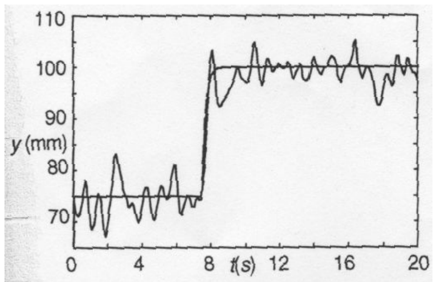
Ball levitation

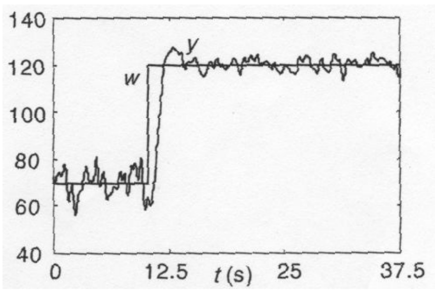


$$G_S(s) = \frac{K\exp(-\tau s)}{Ts + \exp(-\vartheta s)}$$

$$K = 62.5;\ \tau = 0.7s$$

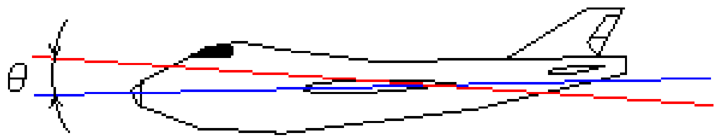
$$\omega_u = 3.5\mathrm{s}^{-1};\ k_u = 0.02 \Rightarrow T = 0.31s;\ \vartheta = 0.08s$$





Examples of applications

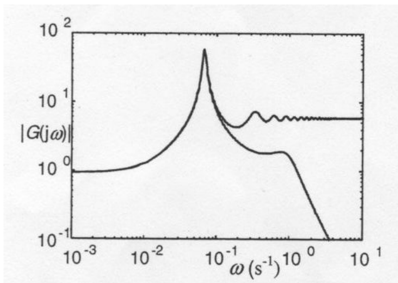
Pitch attitude

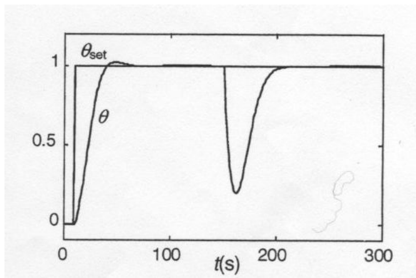


$$G_\theta(s) = \frac{-\left(b_2 s^2 + b_1 s + b_0\right)}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}; b_0 = 3.9\cdot 10^{-3}, b_1 = 0.3545, b_2 = 1.158,$$

$$a_0 = 4.2\cdot 10^{-3}, a_1 = 9.5\cdot 10^{-3}, a_2 = 0.9355$$

$$\overline{G}_\theta(s) = \frac{Ps + \exp(-\psi s)}{Ts + \exp(-\vartheta s)}\exp(-\tau s); P = 101, \psi = 11.51, T = 16.64, \vartheta = 23.4, \tau = 5$$





Conclusions

- Modeling of continuous-time systems
  - Less state variables
  - Easy to identify
  - Describe “real” plant behavior
- Possibility to use algebraic control methods
  - IMC
  - R_{MS}
- Many potential applications
  - Transport of mass or energy

Thank you for your attention