# Efficient Software Tools for Control and Analysis of Hybrid Systems


**Michal Kvasnica**

Automatic Control Laboratory, ETH Zürich

www.control.ethz.ch


# Outline

1. Problem statement

2. Efficient computation and evaluation of feedback controllers

3. Software tools in theory and practice

4. Conclusions

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# Piecewise Affine (PWA) Systems

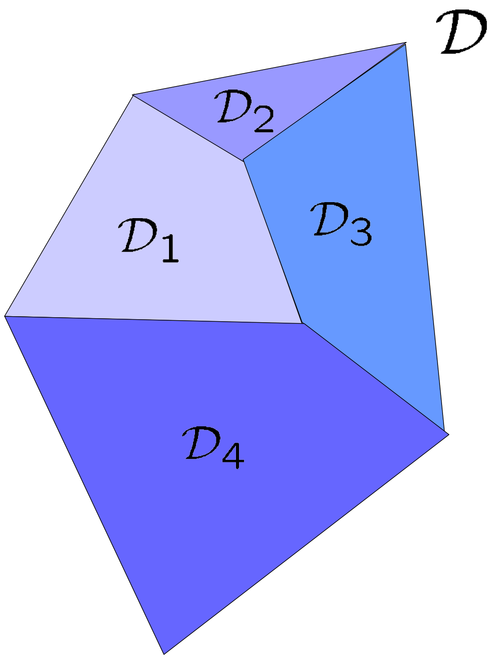
$$\begin{aligned} x(t+1) &= A_i x(t) + B_i u(t) + f_i + w(t) \\ y(t) &= C_i x(t) + D_i u(t) + g_i \\ G^x x(t) &+ G^u u(t) \leq G^c \end{aligned} \quad \text{if} \quad \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}_i$$

- Equivalent to many classes of hybrid systems

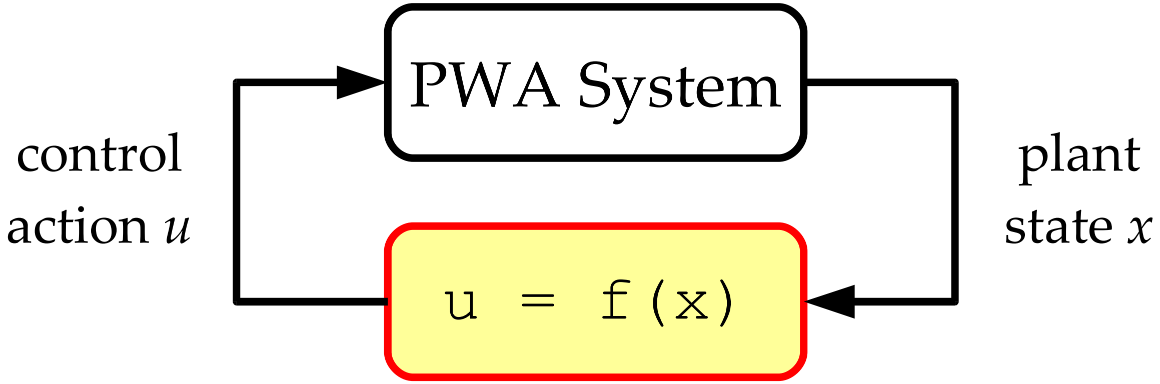
  *(Heemels, De Schutter, Bemporad, 2001)*

- Include constrained linear discrete time systems

- Can approximate general non-linear systems with arbitrary precision



# Control Objectives




- Stability (feedback is stabilizing)
- Feasibility (feedback exists for all time)
- Optimal performance
- Implementable in real time

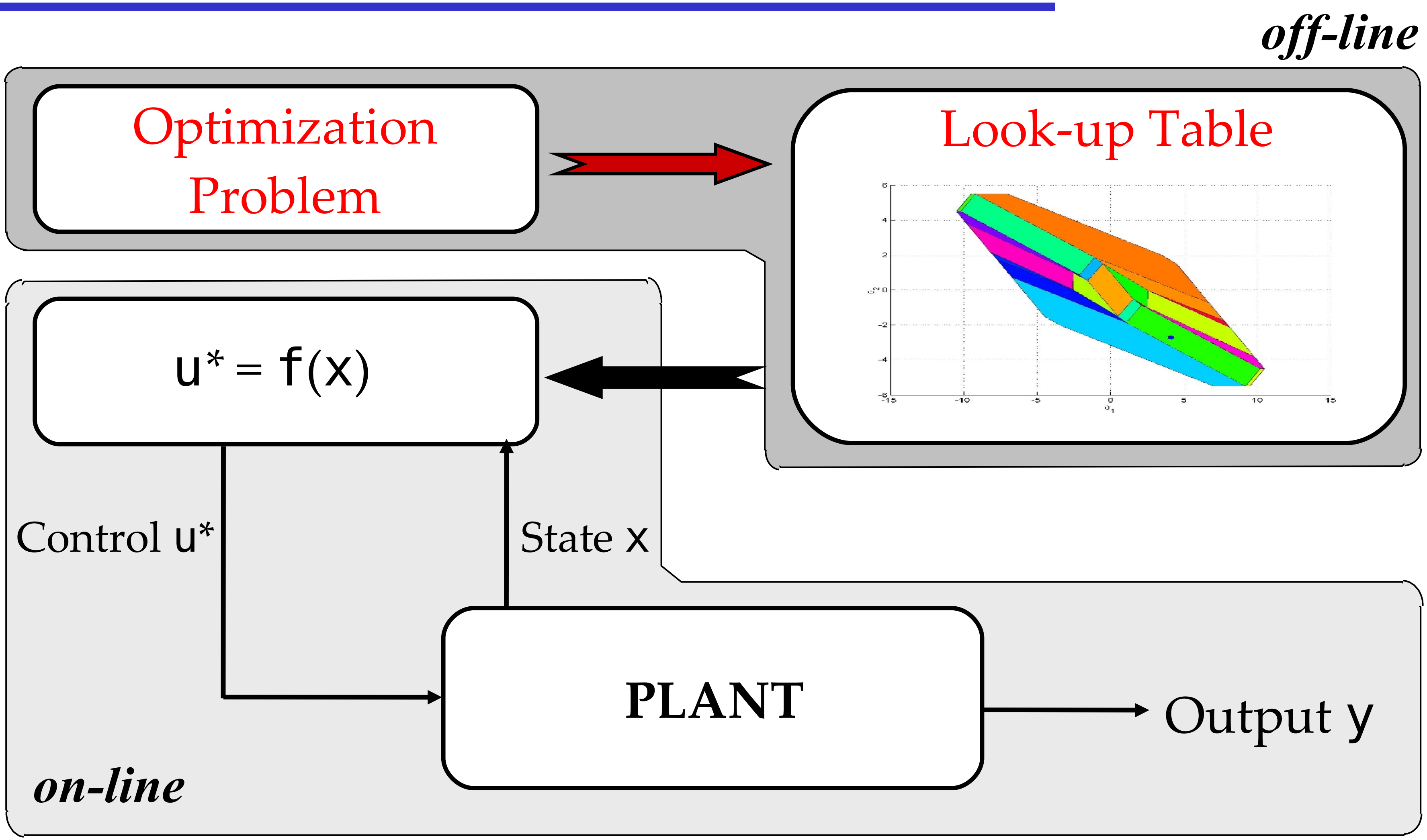
# Optimal Control of Constrained Systems

- Formulate a Constrained Finite Time Optimal Control (CFTOC) problem:

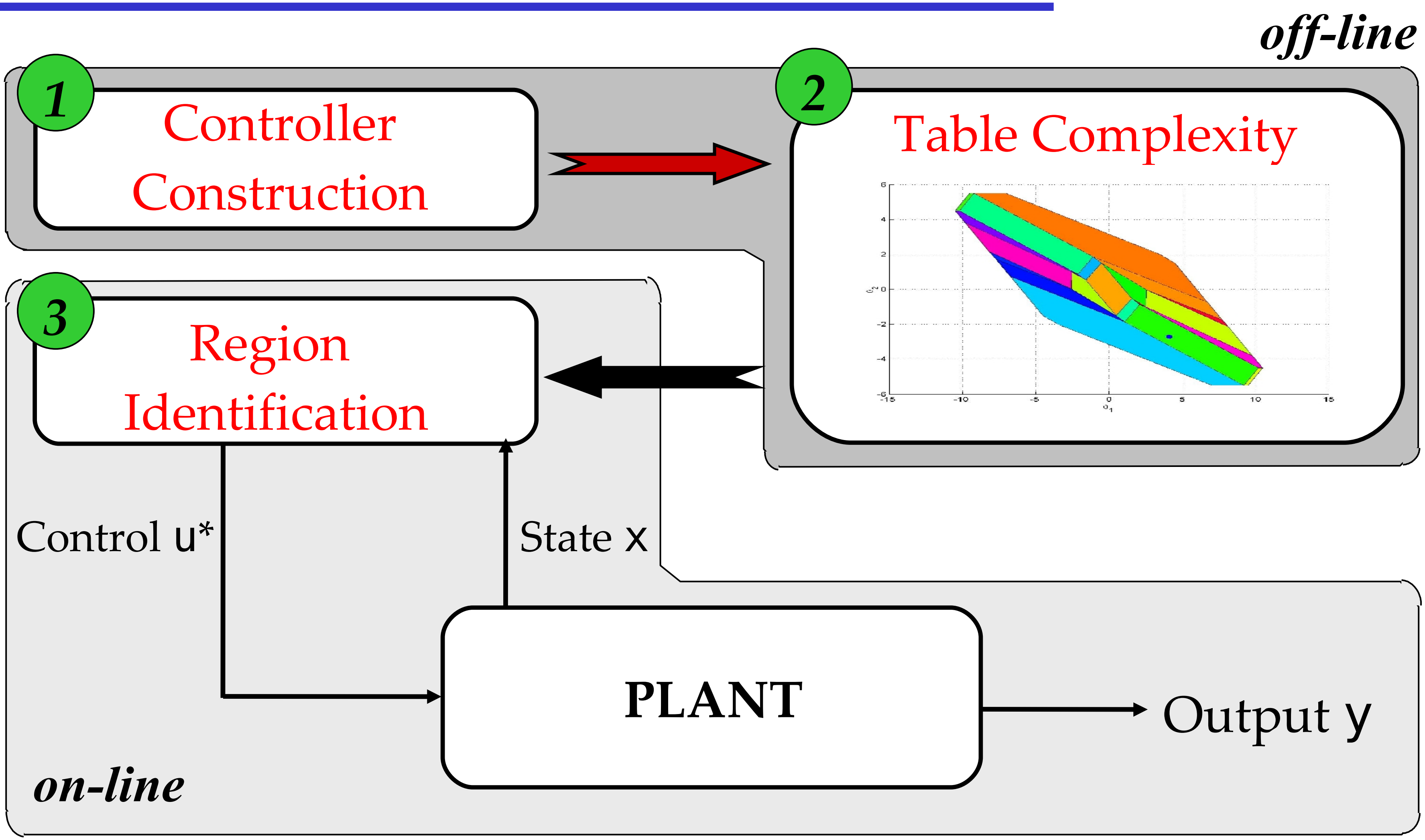
$$
\begin{aligned}
J_N^*(x(0)) \;&=\; \min_{u_0,\ldots,u_{N-1}} \left\{ \sum_{k=0}^{N-1} \left( u_k' \mathcal{R} u_k + x_k' \mathcal{Q} x_k \right) + x_N' \mathcal{Q}_f x_N \right\}, \\
\text{subj. to} \quad & x_k \in \mathbb{X}, \qquad k \in \{0, \ldots, N\}, \\
& u_k \in \mathbb{U}, \qquad k \in \{0, \ldots, N-1\}, \\
& x_{k+1} = f_{PWA}(x_k, u_k), \\
& \mathcal{Q} \succeq 0, \qquad \mathcal{Q}_f \succeq 0, \qquad \mathcal{R} \succ 0.
\end{aligned}
$$

- Compute a solution to the CFTOC problem:
  - on-line for one given initial condition
  - off-line for all admissible initial conditions by applying multi-parametric programming

# Receding Horizon Control (RHC)



# 3 Bottlenecks of Parametric RHC

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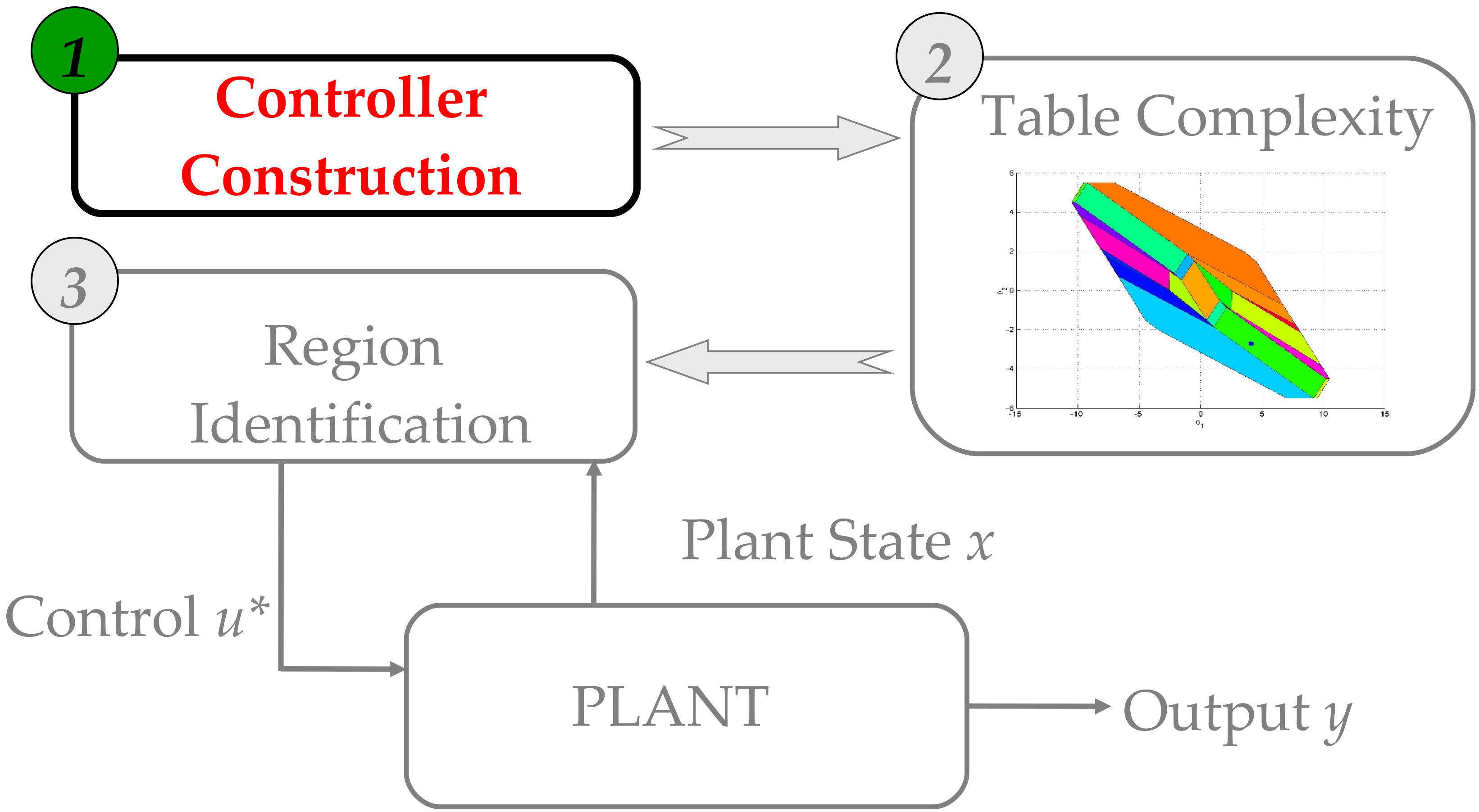
# The 1st Bottleneck

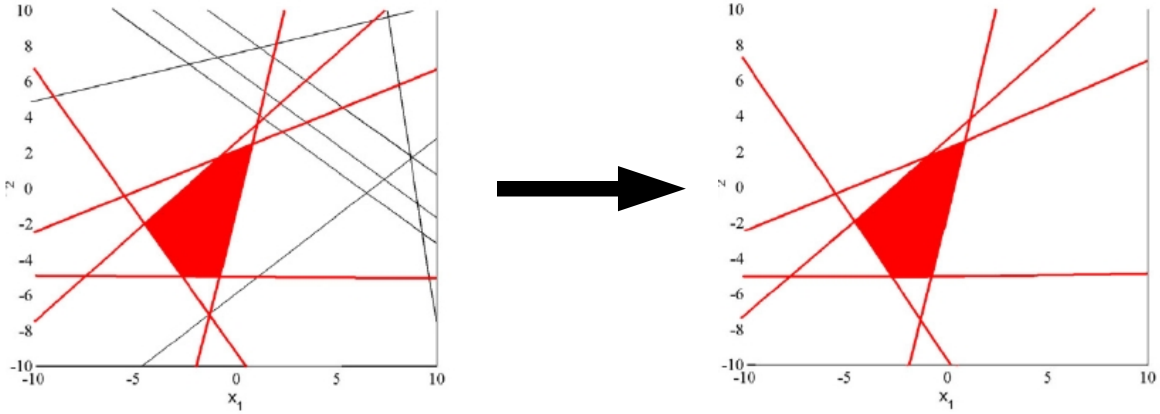


# Controller Construction – Key Runtime Drivers

**Solution time of a multi-parametric program:**

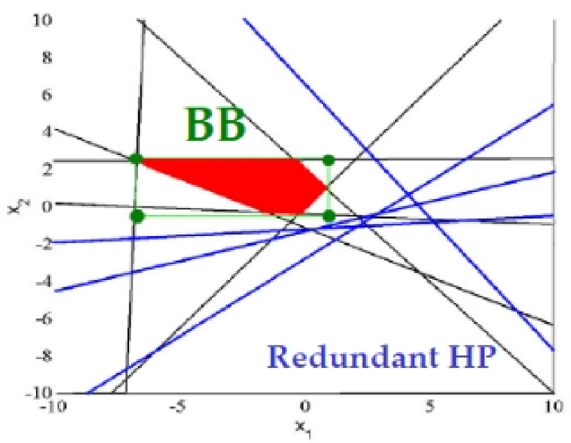
**Runtime per region** $\times$ **# of regions**

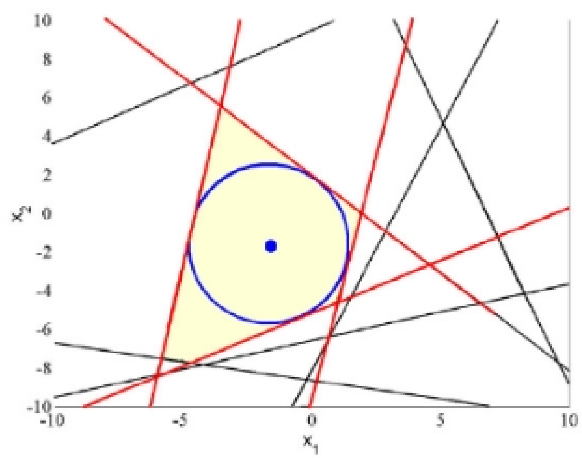
| 60% of the runtime consumed by removal of redundant constraints | The 2nd bottleneck |
|---|---|



# Efficient Redundancy Elimination

- Efficient elimination of redundant constraints using pre-solve techniques to speed up the computation
  (using bounding boxes and chebychev balls)

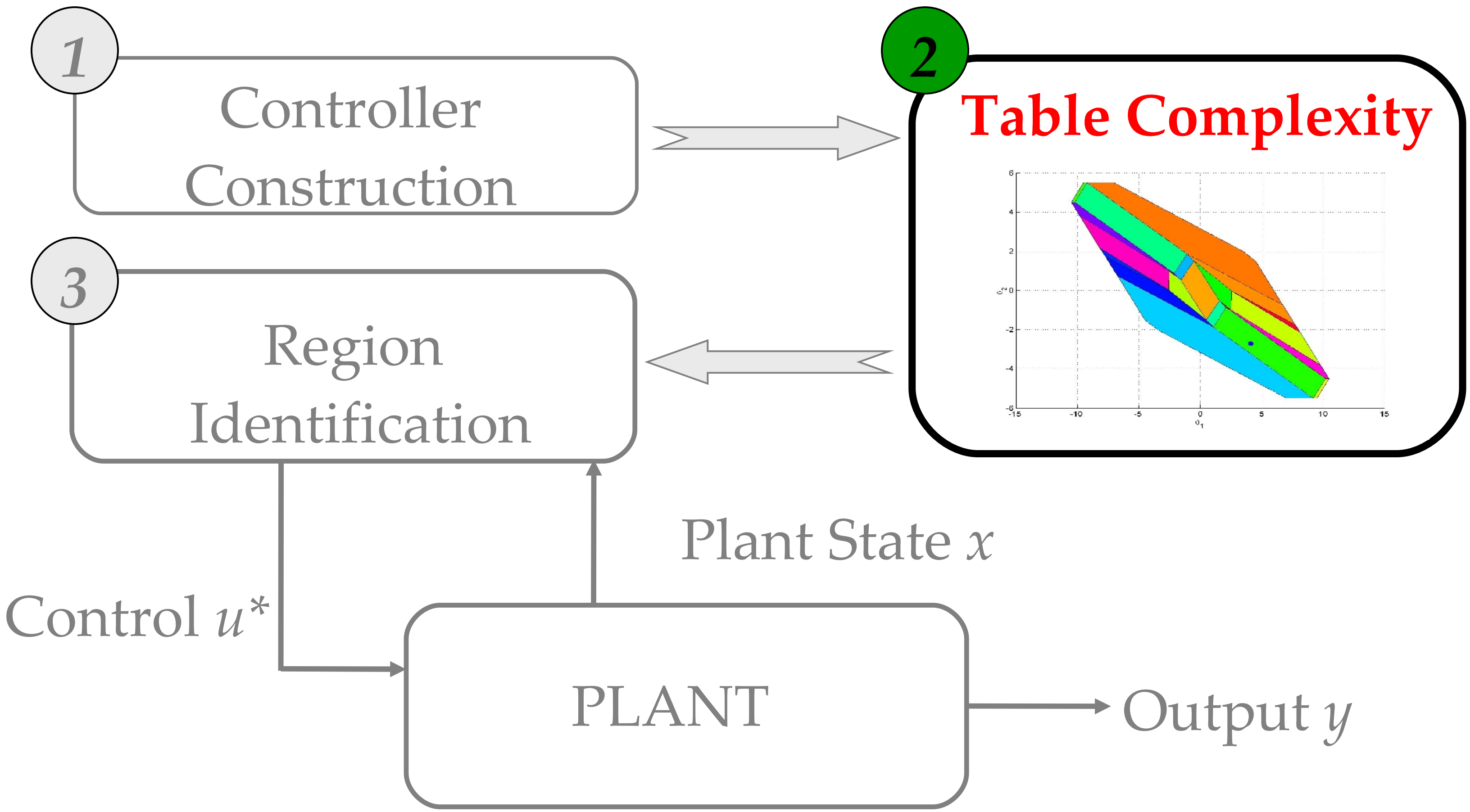


**Result: runtime per region reduced by 50%**

*(Suard, Loefberg, Grieder, Kvasnica, Morari; CDC 04)*

# The 2nd Bottleneck



# Addressing the $2^{nd}$ Bottleneck

## Control objectives

- Stability
- Feasibility
- Optimal performance

## Observation

Complex objectives yield complex controllers

## Approach

Use simpler objectives to obtain simpler controllers

# Minimum-Time Controller

- Specify "simpler" performance objective:
  - Drive state into target set in minimum-time
  - Instead of solving *one* problem of size *N*, solve *N* problems of size *one*
- Stability and constraint satisfaction are guaranteed by construction

**Result: Fewer controller regions**

**"Fast" construction of control law**

*(Grieder, Kvasnica, Baotic, Morari; Automatica 2005)*

# *M*-step Controller

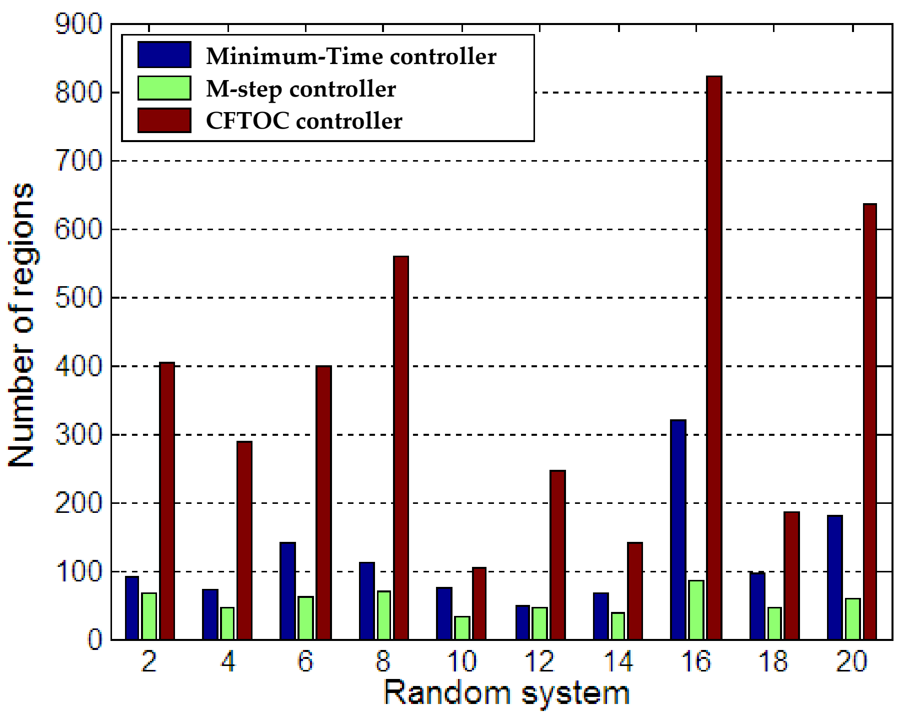
- Do not enforce closed-loop stability:
  - Solve a CFTOC problem for a "short" horizon $M$ with an additional invariant set constraint on $x_1$
- Constraint satisfaction and optimal performance are guaranteed by construction
- Analyse stability of the closed-loop system

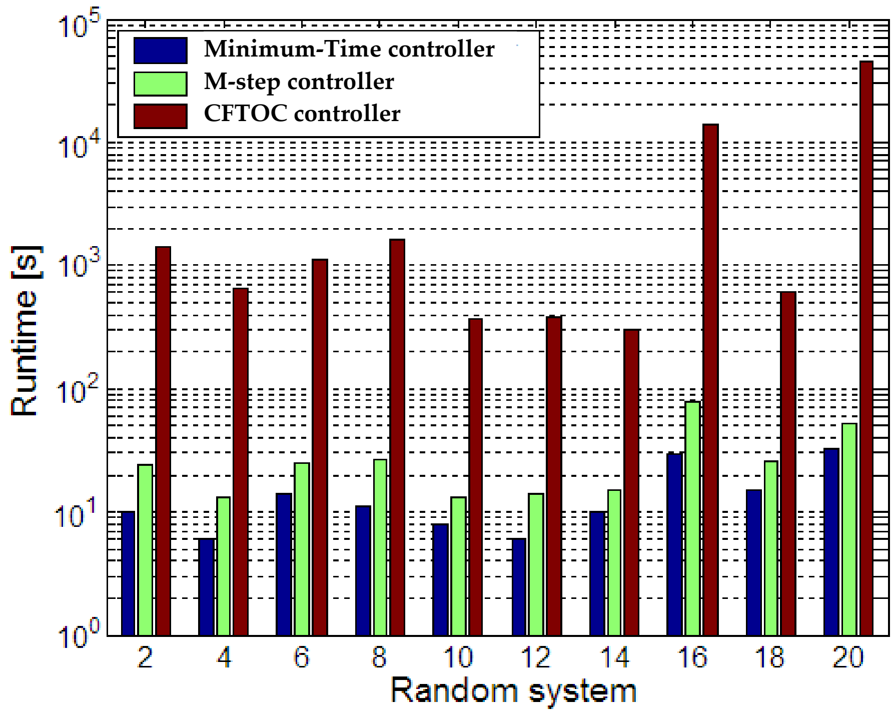
**Result: Significantly fewer controller regions**
**"Fast" construction of the control law**

*(Grieder, Kvasnica, Baotic, Morari; Automatica 2005)*

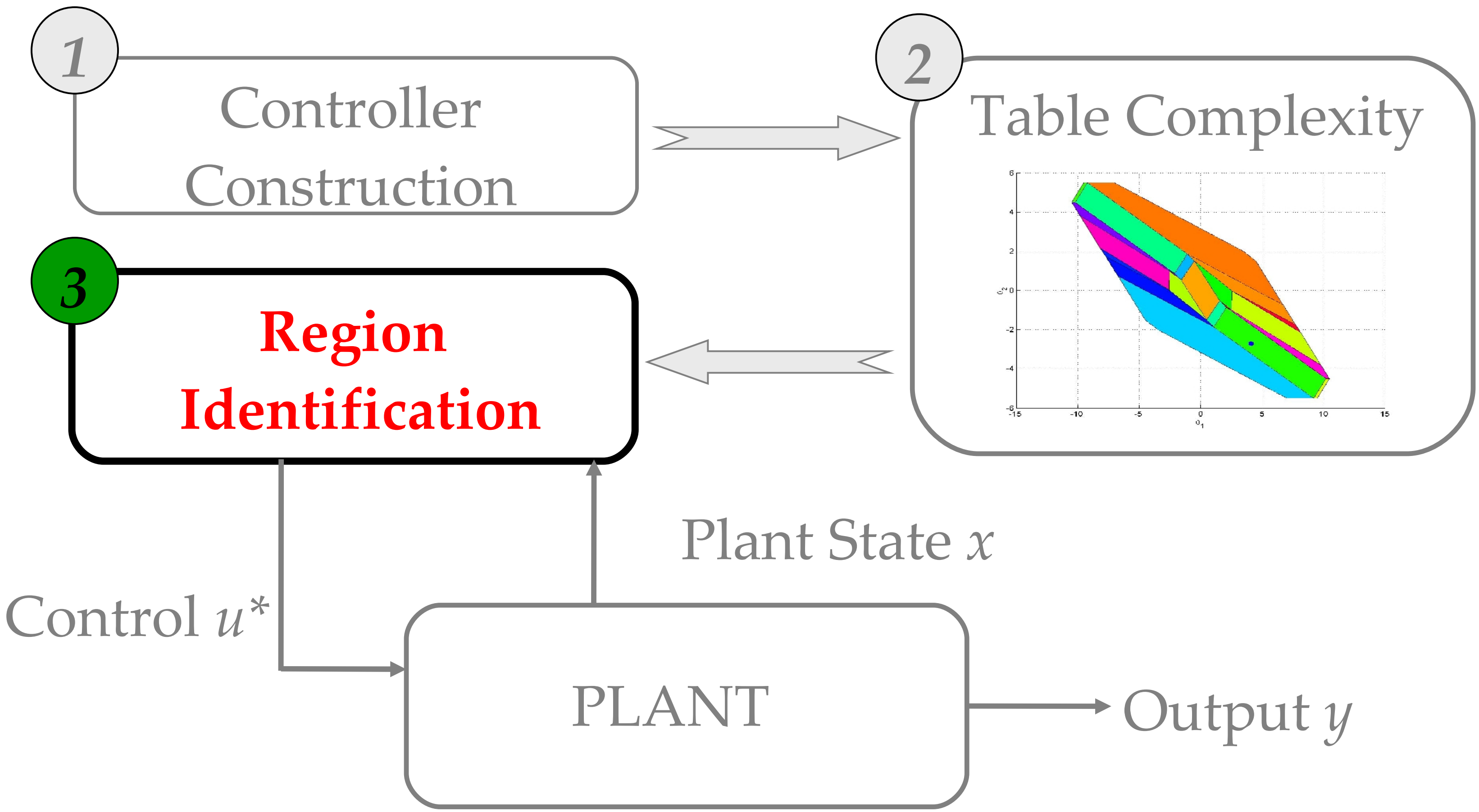
# Numerical Examples

Controllers for **10 random PWA systems** with 2 states, 1 input and 4 different dynamics were computed…



# The 3rd Bottleneck



# Interval Search Trees

## Objective

Search through the look-up table in a fast way

## Idea

Construct an interval search tree based on bounding boxes

## Advantage

- Region identification performed almost in O(log(N)) time
- Very cheap pre-processing compared to other techniques
- Applicable to any type of partitions (not even polyhedral)

*(Christophersen, Kvasnica, Jones, Morari, ECC 2007)*

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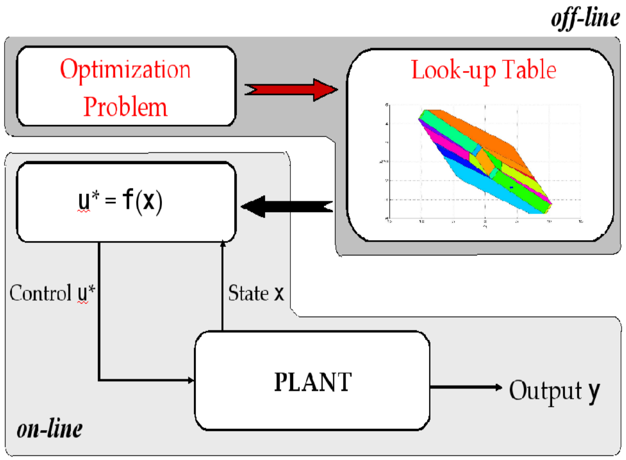
# Theory and Practice

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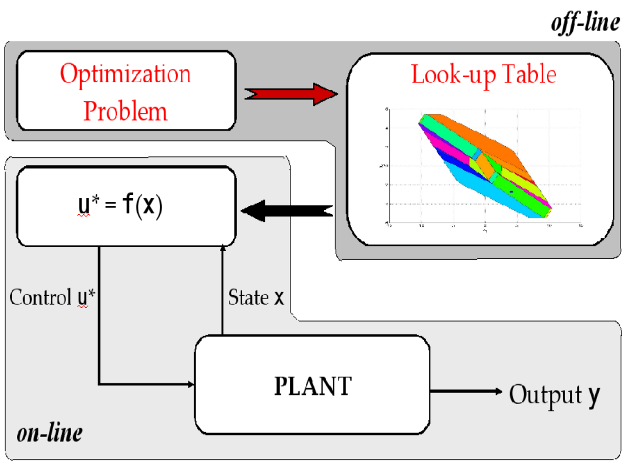
# Multi-Parametric Toolbox (MPT)

## To bridge the gap between theory and practice




MPT

*(Kvasnica, Grieder, Baotic, Morari; HSCC 04)*

# Multi-Parametric Toolbox (MPT)

Model → Control → Analysis → Deployment

MPT is a repository of hybrid systems design tools utilizing state-of-the-art optimization packages

Main strong points:

- Design of low complexity controllers
- Generation of real-time executable code
- Focus on numerical robustness and speed of algorithms
- Released under an open-source GPL license

# MPT: Areas of Applications

- RHC-based control synthesis
- Lyapunov-based stability analysis
- Reachability analysis and safety verification
- Modeling and simulation of hybrid systems
- Computational geometry
- Multi-parametric optimization

# MPT in the World

**11000+** downloads in 4 years

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# Conclusions

- Multi-parametric approach to RHC has many advantages, but also many limitations
- Novel algorithms developed to reduce the complexity by orders of magnitude
- Software tools created to bridge the gap between theory and practice